# DIFFERENTIATING THE ABSOLUTELY CONTINUOUS INVARIANT MEASURE OF AN INTERVAL MAP $f$ WITH RESPECT TO $f$.

by David Ruelle*.

**Abstract.** *Let the map* $f : [-1,1] \to [-1,1]$ *have a.c.i.m.* $\rho$ *(absolutely continuous* $f$*-invariant measure with respect to Lebesgue). Let* $\delta\rho$ *be the change of* $\rho$ *corresponding to a perturbation* $X = \delta f \circ f^{-1}$ *of* $f$*. Formally we have, for differentiable* $A$*,*

$$\delta\rho(A) = \sum_{n=0}^{\infty} \int \rho(dx)\, X(x) \frac{d}{dx} A(f^n x)$$

*but this expression does not converge in general. For* $f$ *real-analytic and Markovian in the sense of covering* $(-1,1)$ $m$ *times, and assuming an* analytic expanding *condition, we show that*

$$\lambda \mapsto \Psi(\lambda) = \sum_{n=0}^{\infty} \lambda^n \int \rho(dx)\, X(x) \frac{d}{dx} A(f^n x)$$

*is meromorphic in* $\mathbf{C}$*, and has no pole at* $\lambda = 1$*. We can thus formally write* $\delta\rho(A) = \Psi(1)$*.*

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We postpone a discussion of the significance of our result, and start to describe the conditions under which we prove it. Note that these conditions are certainly too strong: suitable differentiability should replace analyticity, and a weaker Markov property should be sufficient. But the point of the present note is to show how it is that $\Psi(\lambda)$ has no pole at $\lambda = 1$, rather than deriving a very general theorem.

**Setup.**

We assume that $f : [-1, 1] \to [-1, 1]$ is real analytic and piecewise monotone on $[-1, 1]$ in the following sense: there are points $c_j$ ($j = 0, \ldots, m$, with $m \geq 2$) such that $-1 = c_0 < c_1 < \ldots < c_{m-1} < c_m = 1$ and, for $j = 0, \ldots, m$,

$$f(c_j) = (-1)^{j+1}$$

We assume that on $[-1, 1]$ the derivative $f'$ vanishes only on $Z = \{c_1, \ldots, c_{m-1}\}$, and that $f''$ does not vanish on $Z$. For $j = 1, \ldots, m$, we have $f[c_{j-1}, c_j] = [-1, 1]$. In particular, $f$ is Markovian. We shall also assume that $f$ is *analytically expanding* in the sense of Assumption A below. The purpose of this note is to prove the following:

**Theorem.** *Under the above conditions, and Assumption A stated later, there is a unique $f$-invariant probability measure $\rho$ absolutely continuous with respect to Lebesgue on $[-1, 1]$. If $X$ is real-analytic on $[-1, 1]$, and $A \in \mathcal{C}^1[-1, 1]$, then*

$$\Psi(\lambda) = \sum_{n=0}^{\infty} \lambda^n \int_{-1}^{1} \rho(dx)\, X(x) \frac{d}{dx} A(f^n x)$$

*extends to a meromorphic function in $\mathbf{C}$, without pole at $\lambda = 1$.*

Our proof depends on a change of variable which we now explain. We choose a holomorphic function $\omega$ from a small open neighborhood $U_0$ of $[-1, 1]$ in $\mathbf{C}$ to a small open neighborhood $W$ of $[-1, 1]$ in a Riemann surface which is 2-sheeted over $\mathbf{C}$ near $-1$ and 1. We call $\varpi = \omega^{-1} : W \to U_0$ the inverse of $\omega$. We assume that $\omega(-x) = -\omega(x)$, $\omega(\pm 1) = \pm 1$, $\omega[-1, 1] = [-1, 1]$, $\omega'(\pm 1) = \omega'''(\pm 1) = 0$. We have thus

$$\omega(\pm(1 - \xi)) = \pm(1 - C\xi^2 + D\xi^4 \ldots)$$

with $C > 0$ and, if $a > 0$,

$$\varpi(\pm(1 - a\xi^2 + b\xi^3 \ldots)) = \pm(1 - \sqrt{\frac{a}{C}}\xi + \frac{b}{2\sqrt{aC}}\xi^2 \ldots)$$

[We may for instance take

$$\omega(x) = \sin \frac{\pi x}{2} \qquad , \qquad \varpi(x) = \frac{2}{\pi} \arcsin x$$

or

$$\omega(x) = \frac{1}{16}(25x - 10x^3 + x^5) \qquad , \qquad \varpi(x) = \frac{16}{25} x \ldots \qquad ]$$

The function $g : \varpi \circ f \circ \omega$ from $[-1,1]$ to $[-1,1]$ has monotone restrictions to the intervals $\varpi[c_{j-1},c_j] = [d_{j-1},d_j]$. It is readily seen that $g_j$ extends to a holomorphic function in a neighborhood of $[d_{j-1},d_j]$, and that

$$g_1(-1+\xi) = -1 + \sqrt{f'(-1)}\xi + \alpha_-\xi^3 \ldots$$

$$g_m(1-\xi) = (-1)^{m+1}(1 - \sqrt{|f'(1)|}\xi - \alpha_+\xi^3 \ldots)$$

with no $\xi^2$ terms in the right-hand sides [this follows from our choice of $\omega$, which has no $\xi^3$ term]. One also finds that, for $j = 1, \ldots, m-1$

$$g_j(d_j - \xi) = (-1)^{j+1}(1 - \sqrt{\frac{|f''(c_j)|}{2C}}\omega'(d_j)\xi + \gamma_j\xi^2 \ldots)$$

$$g_{j+1}(d_j + \xi) = (-1)^{j+1}(1 - \sqrt{\frac{|f''(c_j)|}{2C}}\omega'(d_j)\xi - \gamma_j\xi^2 \ldots)$$

where $\gamma_j$ is the same in the two relations. We note the following easy consequences of the above developments:

**Lemma 1.** *Let $\psi_j : [-1,1] \to [d_{j-1},d_j]$ be the inverse of $g_j$ for $j = 1, \ldots, m$ (increasing for $j$ odd, decreasing for $j$ even). Then*

$$\psi_1(-1+\xi) = -1 + \frac{1}{\sqrt{f'(-1)}}\xi + \beta_-\xi^3$$

$$\psi_m((-1)^{m+1}(1-\xi)) = 1 - \frac{1}{\sqrt{|f'(1)|}}\xi + \beta_+\xi^3$$

*(there are no $\xi^2$ terms in the right-hand sides). If $j < m$,*

$$\psi_j((-1)^{j+1}(1-\xi)) = d_j - \sqrt{\frac{2C}{|f''(c_j)|}}\frac{1}{\omega'(d_j)}\xi + \delta_j\xi^2$$

$$\psi_{j+1}((-1)^{j+1}(1-\xi)) = d_j + \sqrt{\frac{2C}{|f''(c_j)|}}\frac{1}{\omega'(d_j)}\xi + \delta_j\xi^2$$

*(with the same coefficient $\delta_j$).* $\square$

As inverses of the $g_j$, the functions $\psi_j$ extend to holomorphic functions on a neighborhood of $[-1,1]$. We impose now the condition that $f$ is *analytically expanding* in the following sense:

**Assumption A** *We have $[-1,1] \subset U \subset \mathbf{C}$, with $U$ bounded open connected, such that the $\psi_j$ extend to continuous functions $\bar{U} \mapsto \mathbf{C}$, holomorphic in $U$, and with $\psi_j\bar{U} \subset U$. [$\bar{U}$ denotes the closure of $U$].*

Let $\phi$ be holomorphic on a neighborhood of $\bar{U}$. Given a sequence $\mathbf{j} = (j_1, \ldots, j_\ell, \ldots)$ we define $\phi_{\mathbf{j}\ell} = \phi \circ \psi_{j_1} \cdots \circ \psi_{j_\ell}$ and note that the $\phi_{\mathbf{j}\ell}$ are uniformly bounded in a neighborhood of $\bar{U}$. We may thus choose $\ell(r)$ for $r = 1, 2 \ldots$ such that the subsequence $(\phi_{\mathbf{j}\ell(r)})_{r=1}^{\infty}$ converges uniformly on $\bar{U}$ to a limit $\tilde{\phi}_{\mathbf{j}}$. Writing $\tilde{U} = \cup_{j=1}^{m} \psi_j \bar{U}$ we have

$$\max_{z \in \bar{U}} |\phi_{\mathbf{j}\ell(r)}| \geq \max_{z \in \tilde{U}} |\phi_{\mathbf{j}\ell(r)}| \geq \max_{z \in \bar{U}} |\phi_{\mathbf{j}\ell(r+1)}|$$

so that $\max_{z \in \bar{U}} |\tilde{\phi}_{\mathbf{j}}| = \max_{z \in \tilde{U}} |\tilde{\phi}_{\mathbf{j}}|$ and, since $\tilde{U}$ is compact $\subset U$ connected, $\tilde{\phi}_{\mathbf{j}}$ is constant. Therefore $\phi$ is constant on $\cap_{\ell=0}^{\infty} \psi_{j_1} \circ \cdots \circ \psi_{j_\ell} \bar{U}$. Since this is true for all $\phi$, the intersection $\cap_{\ell=0}^{\infty} \psi_{j_1} \circ \cdots \circ \psi_{j_\ell} \bar{U}$ consists of a single point $\tilde{z}(\mathbf{j})$. Given $\epsilon > 0$ we can thus, for each $\mathbf{j}$, find $\ell$ such that $\mathrm{diam} \psi_{j_1} \circ \cdots \circ \psi_{j_\ell} \bar{U} < \epsilon$. Hence (using the compactness of the Cantor set of sequences $\mathbf{j}$) one can choose $L$ so that the $m^L$ sets

$$\psi_{j_1} \circ \cdots \psi_{j_L} \bar{U}$$

have diameter $< \epsilon$. The open connected set

$$V = \cup_{j_1, \ldots, j_L} \psi_{j_1} \circ \cdots \psi_{j_L} U$$

satisfies $[-1, 1] \subset V \subset U$, and $\psi_j \bar{V} = \cup_{j_1, \ldots, j_L} \psi_j \circ \psi_{j_1} \circ \cdots \circ \psi_{j_L} \bar{U} \subset \cup_{j_0, j_1, \ldots, j_{L-1}} \psi_{j_0} \circ \psi_{j_1} \circ \psi_{i_{L-1}} U = V$. This shows that $U$ can be replaced in Assumption A by a set $V$ contained in an $\epsilon$-neighborhood of $[-1, 1]$.

Since we have shown above that $\mathrm{diam} \psi_{j_1} \circ \cdots \psi_{j_L} \bar{U} < \epsilon$, we see that $\psi_1^L$ maps a small circle around $-1$ strictly inside itself. We have thus $\psi_1'(-1) < 1$ (*i.e.*, $f'(-1) > 1$) and similarly, if $m$ is odd, $\psi_m'(1) < 1$ (*i.e.*, $f'(1) > 1$).

The following two lemmas state some easy facts to be used later.

**Lemma 2.** *Let $H$ be the Hilbert space of functions $\bar{U} \to \mathbf{C}$ which are square integrable (with respect to Lebesgue) and holomorphic in $U$. The operator $\mathcal{L}$ on $H$ defined by*

$$(\mathcal{L}\Phi)(z) = \sum_{j=1}^{m} (-1)^{j+1} \psi_j'(z) \Phi(\psi_j(z))$$

*is holomorphy improving. In particular $\mathcal{L}$ is compact and trace-class.* $\square$

**Lemma 3.** *On $[-1, 1]$ we have*

$$(\mathcal{L}\Phi)(x) = \sum_j |\psi_j'(x)| \Phi(\psi_j(x))$$

*hence $\Phi \geq 0$ implies $\mathcal{L}\Phi \geq 0$ ($\mathcal{L}$ preserves positivity) and*

$$\int_{-1}^{1} dx \, (\mathcal{L}\Phi)(x) = \int_{-1}^{1} dx \, \Phi(x)$$

($\mathcal{L}$ *preserves total mass*). $\square$

**Lemma 4.** *$\mathcal{L}$ has a simple eigenvalue $\mu_0 = 1$ corresponding to an eigenfunction $\sigma_0 > 0$. The other eigenvalues $\mu_k$ ($k \geq 1$) satisfy $|\mu_k| < 1$, and their (generalized) eigenfunctions $\sigma_k$ satisfy $\int_{-1}^{1} dx\, \sigma_k(x) = 0$.*

Let $(\mu_k, \sigma_k)$ be a listing of the eigenvalues and generalized eigenfunctions of the trace-class operator $\mathcal{L}$. For each $\mu_k$ there is some $\sigma_k$ such that $\mathcal{L}\sigma_k = \mu_k \sigma_k$, hence

$$|\mu_k| \int_{-1}^{1} dx\, |\sigma_k(x)| = \int_{-1}^{1} dx\, |\mu_k \sigma_k(x)| = \int_{-1}^{1} dx\, |(\mathcal{L}\sigma_k)(x)|$$

$$\leq \int_{-1}^{1} dx\, (\mathcal{L}|\sigma_k|)(x) = \int_{-1}^{1} dx\, |\sigma_k(x)|$$

hence $|\mu_k| \leq 1$. Denote by $S_<$ and $S_1$ the spectral spaces of $\mathcal{L}$ corresponding to eigenvalues $\mu_k$ with $|\mu_k| < 1$, and $|\mu_k| = 1$ respectively. If $\sigma_k \in S_<$ then, for some $n \geq 1$,

$$0 = \int_{-1}^{1} dx\, ((\mathcal{L} - \mu_k)^n \sigma_k)(x) = \int_{-1}^{1} dx\, (1 - \mu_k)^n \sigma_k(x)$$

hence $\int_{-1}^{1} dx\, \sigma_k(x) = 0$.

On the finite dimensional space $S_1$, there is a basis of eigenvectors $\sigma_k$ diagonalizing $\mathcal{L}$ (if $\mathcal{L}|S_1$ had non-diagonal normal form, $||\mathcal{L}^n|S_1||$ would tend to infinity with $n$, in contradiction with $\int_{-1}^{1} dx\, |(\mathcal{L}^n\Phi)(x)| \leq \int_{-1}^{1} dx\, |\Phi(x)|$). We shall now show that, up to multiplication by a constant $\neq 0$, we may assume $\sigma_k \geq 0$. If not, because $\sigma_k$ is continuous and the intervals $\psi_{j_1} \circ \cdots \circ \psi_{j_n}[-1, 1]$ are small for large $n$ (mixing), we would have $|(\mathcal{L}^n\sigma_k)(x)| < (\mathcal{L}^n|\sigma_k|)(x)$ for some $n$ and $x$. This would imply $\int_{-1}^{1} dx\, |(\mathcal{L}^n\sigma_k)(x)| < \int_{-1}^{1} dx\, |\sigma_k(x)|$ in contradiction with $\mathcal{L}\sigma_k = \mu_k\sigma_k$ and $|\mu_k| = 1$. From $\sigma_k \geq 0$ we get $\mu_k = 1$, and the corresponding eigenspace is at most one dimensional (otherwise it would contain functions not $\geq 0$). But we have $1 \notin S_<$ because $\int_{-1}^{1} dx, 1 \neq 0$, so that $S_1 \neq \{0\}$. Thus $S_1$ is spanned by an eigenfunction, which we call $\sigma_0$, to the eigenvalue $\mu_0 = 1$. Finally, $\sigma_0 > 0$ because if $\sigma_0(x) = 0$ we would have also $\sigma_0(y) = 0$ whenever $g^n(y) = x$, which is not compatible with $\sigma_0$ continuous $\neq 0$. $\square$

**Lemma 5.** *If we normalize $\sigma_0$ by $\int_{-1}^{1} dx\, \sigma_0(x) = 1$, then $\sigma_0(dx) = \sigma_0(x)dx$ is the unique $g$-invariant probability measure absolutely continuous with respect to Lebesgue on $[-1, 1]$. In particular, $\sigma_0(dx)$ is ergodic.*

For continuous $A$ on $[-1, 1]$ we have

$$\int_{-1}^{1} \sigma_0(dx)(A \circ g)(x) = \int_{-1}^{1} dx\, \sigma_0(x) A(g(x)) = \int_{-1}^{1} dx\, (\mathcal{L}\sigma_0)(x) A(x) = \int_{-1}^{1} \sigma_0(dx) A(x)$$

so that $\sigma_0(dx)$ is $g$-invariant. Let $\tilde{\sigma}(x)dx$ be another $g$-invariant probability measure absolutely invariant with respect to Lebesgue. Then, if $\tilde{\sigma} \neq \sigma_0$

$$\int_{-1}^{1} dx\, |\sigma_0(x) - \tilde{\sigma}(x)| = \int_{-1}^{1} dx\, |(\mathcal{L}(\sigma_0 - \tilde{\sigma}))(x)|$$

$$< \int_{-1}^{1} dx\, (\mathcal{L}|\sigma_0 - \tilde{\sigma}|)(x) = \int_{-1}^{1} dx\, |\sigma_0(x) - \tilde{\sigma}(x)|$$

by mixing: contradiction. ∎

**Lemma 6.** *Let $H_1 \subset H$ consist of those functions $\Phi$ with derivatives vanishing at $\pm 1$: $\Phi'(-1) = \Phi'(1) = 0$. Then $\mathcal{L}H_1 \subset H_1$ and $\sigma_0 \in H_1$.*

$\mathcal{L}H_1 \subset H_1$ is an easy calculation using Lemma 1. Furthermore, by Lemma 4, $\sigma_0 = \lim_{n\to\infty} \mathcal{L}^n \frac{1}{2}$, and $\frac{1}{2} \in H_1$ implies $\sigma_0 \in H_1$. ∎

The image $\rho(dx) = \rho(x)dx$ of $\sigma_0(x)dx$ by $\omega$ is the unique $f$-invariant probability measure absolutely continuous with respect to Lebesgue on $[-1, 1]$. We have

$$\rho(x) = \sigma_0(\varpi x)\varpi'(x)$$

Consider now the expression

$$\Psi(\lambda) = \sum_{n=0}^{\infty} \lambda^n \int_{-1}^{1} \rho(dx)\, X(x) \frac{d}{dx} A(f^n x)$$

where we assume that $X$ extends to a holomorphic function in a neighborhood of $[-1, 1]$ and $A \in \mathcal{C}^1[-1, 1]$. For sufficiently small $|\lambda|$, the series defining $\Psi(\lambda)$ converges. Writing $B = A \circ \omega$ and $x = \omega y$ we have

$$X(x) \frac{d}{dx} A(f^n x) = X(\omega y) \frac{1}{\omega'(y)} \frac{d}{dy} B(g^n y)$$

hence

$$\Psi(\lambda) = \sum_{n=0}^{\infty} \lambda^n \int_{-1}^{1} dy\, \sigma_0(y) \frac{X(\omega y)}{\omega'(y)} \frac{d}{dy} B(g^n y)$$

Defining $Y(y) = \sigma_0(y) X(\omega y)/\omega'(y)$, we see that $Y$ extends to a function holomorphic in a neighborhood of $[-1, 1]$, which we may take to be $U$, except for simple poles at $-1$ and $1$. We may write

$$\int_{-1}^{1} dy\, \sigma_0(y) \frac{X(\omega y)}{\omega'(y)} \frac{d}{dy} B(g^n y) = \int_{-1}^{1} dy\, Y(y) g'(y) \cdots g'(g^{n-1} y) B'(g^n y)$$

$$= \int_{-1}^{1} ds\, (\mathcal{L}_0^n Y)(s) B'(s)$$

where

$$(\mathcal{L}_0 \Phi)(s) = \sum_{j=1}^{m} (-1)^{j+1} \Phi(\psi_j s)$$

and we have thus

$$\Psi(\lambda) = \sum_{n=0}^{\infty} \lambda^n \int_{-1}^{1} ds\, (\mathcal{L}_0^n Y)(s) B'(s)$$

**Lemma 7.** *Let $H_0 \subset H$ be the space of functions vanishing at $-1$ and $1$. Then $\mathcal{L}_0 H_0 \subset H_0$.*

This follows readily from Lemma 1. $\square$

**Lemma 8.** *There are meromorphic functions $\Phi_\pm$ with Laurent series*

$$\Phi_\pm(z) = \frac{1}{z \mp 1} + O(z \mp 1)$$

*at $\pm 1$ and $\Phi_\pm(\mp 1) = 0$ such that*

$$\mathcal{L}_0 \Phi_- = \sqrt{f'(-1)}\Phi_-$$

$$\begin{cases} \mathcal{L}_0 \Phi_+ = \sqrt{f'(1)}\Phi_+ & \text{if } m \text{ is odd} \\ \mathcal{L}_0(\Phi_+/\sqrt{|f'(1)|} + \Phi_-/\sqrt{f'(-1)}) = \tilde{Y} \in H_0 & \text{if } m \text{ is even} \end{cases}$$

Define

$$p_\pm(z) = \frac{1}{z \mp 1} - \frac{1}{4}(z \mp 1)$$

then Lemma 1 yields

$$(\mathcal{L}_0 - \sqrt{f'(-1)})p_- = u_- \in H_0$$

$$\begin{cases} (\mathcal{L}_0 - \sqrt{f'(1)})p_+ = u_+ \in H_0 & \text{if } m \text{ is odd} \\ \mathcal{L}_0 p_+ + \sqrt{|f'(1)|}p_- = u_0 \in H_0 & \text{if } m \text{ is even} \end{cases}$$

Since $f'(-1) > 1$, Lemma 4 shows that $\mathcal{L} - \sqrt{f'(-1)}$ is invertible on $H$, hence there is $v_-$ such that

$$(\mathcal{L} - \sqrt{f'(-1)})v_- = u'_-$$

and since $\int_{-1}^{1} dx\, u'_-(x) = 0$, also $\int_{-1}^{1} dx\, v_-(x) = 0$ and we can take $w_- \in H_0$ such that $w'_- = v_-$. Then

$$((\mathcal{L}_0 - \sqrt{f'(-1)})w_-)' = (\mathcal{L} - \sqrt{f'(-1)})w'_- = (\mathcal{L} - \sqrt{f'(-1)})v_- = u'_-$$

so that

$$(\mathcal{L}_0 - \sqrt{f'(-1)})w_- = u_-$$

without additive constant because the left-hand side is in $H_0$ by Lemma 7. In conclusion

$$(\mathcal{L}_0 - \sqrt{f'(-1)})(p_- - w_-) = 0$$

and we may take $\Phi_- = p_- - w_-$.

If $m$ is odd, $\Phi_+$ is handled similarly. If $m$ is even, taking $\Phi_+ = p_+$ and writing $\tilde{Y} = u_0/\sqrt{|f'(1)|} - w_-$ we obtain

$$\mathcal{L}_0(\frac{\Phi_+}{\sqrt{|f'(1)|}} + \frac{\Phi_-}{\sqrt{f'(-1)}}) = \tilde{Y} \in H_0$$

which completes the proof. $\square$

We have $\sigma_0 \in H_1$ (Lemma 6), and $X \circ \omega \in H_1$ by our choice of $\omega$. Also

$$\omega'(\pm(1-\xi)) = 2C\xi - 4D\xi^3 \ldots$$

so that

$$Y = \mathbf{C}\Phi_- + \mathbf{C}\Phi_+ + H_0$$

If $m$ is odd let $Y = c_-\Phi_- + c_+\Phi_+ + Y_0$, with $Y_0 \in H_0$. Then

$$\Psi(\lambda) = \frac{c_-}{1-\lambda\sqrt{f'(-1)}}\int_{-1}^{1} ds\,\Phi_-(s)B'(s) + \frac{c_+}{1-\lambda\sqrt{f'(1)}}\int_{-1}^{1} ds\,\Phi_+(s)B'(s) + \Psi_0(\lambda)$$

where $\Psi_0$ is obtained from $\Psi$ when $Y$ is replaced by $Y_0$.

If $m$ is even let $Y = c_-\Phi_- + \tilde{c}(\Phi_+/\sqrt{|f'(1)|} + \Phi_-/\sqrt{f'(-1)}) + Y_0$, with $Y_0 \in H_0$. Then

$$\Psi(\lambda) = \frac{c_-}{1-\lambda\sqrt{f'(-1)}}\int_{-1}^{1} ds\,\Phi_-(s)B'(s) + \tilde{c}\int_{-1}^{1} ds\,(\frac{\Phi_+}{\sqrt{|f'(1)|}} + \frac{\Phi_-}{\sqrt{f'(-1)}})B'(s)$$

$$+\lambda\tilde{\Psi}(\lambda) + \Psi_0(\lambda)$$

where $\tilde{\Psi}(\lambda)$ is obtained from $\Psi$ when $Y$ is replaced by $\tilde{Y}$.

Writing $\mu_\pm = \sqrt{f'(\pm 1)}$ we see that $\Psi(\lambda)$ has two poles at $\mu_\pm^{-1}$ if $m$ is odd, and one pole at $\mu_-^{-1}$ if $m$ is even; the other poles are those of $\Psi_0(\lambda)$ and possibly $\tilde{\Psi}(\lambda)$. Since $Y_0 \in H_0$ and $\mathcal{L}_0 H_0 \subset H_0$, we have

$$\Psi_0(\lambda) = \sum_{n=0}^{\infty}\lambda^n\int_{-1}^{1} ds\,(\mathcal{L}_0^n Y_0)(s)B'(s) = -\sum_{n=0}^{\infty}\lambda^n\int_{-1}^{1} ds\,(\mathcal{L}_0^n Y_0)'(s)B(s)$$

$$= -\sum_{n=0}^{\infty}\lambda^n\int_{-1}^{1} ds\,(\mathcal{L}^n Y_0')(s)B(s)$$

It follows that $\Psi_0(\lambda)$ extends meromorphically to $\mathbf{C}$ with poles at the $\mu_k^{-1}$. We want to show that the residue of the pole at $\mu_0^{-1} = 1$ vanishes . By Lemma 4, $\int_{-1}^{1} dx\,\sigma_k(x) = 0$ for $k \geq 1$. Thus, up to normalization, the coefficient of $\sigma_0$ in the expansion of $Y_0'$ is

$$\int_{-1}^{1} dx\,Y_0'(x) = Y_0(1) - Y_0(-1) = 0$$

because $Y_0 \in H_0$. Therefore $\Psi_0(z)$ is holomorphic at $z = 1$, and the same argument applies to $\tilde{\Psi}(z)$, concluding the proof of the theorem. $\square$

**Discussion.**

It can be argued that the *physical measure* describing a physical dynamical system is an SRB (Sinai-Ruelle-Bowen) measure $\rho$ (see the recent reviews [11], [2] which contain a number of references), or an a.c.i.m. $\rho$ in the case of a map of the interval. But, typically, physical systems depend on parameters, and it is desirable to know how $\rho$ depends on the parameters (*i.e.*, on the dynamical system). The dependence is smooth for uniformly hyperbolic dynamical systems (see [5], [6] and references given there), but discontinuous in general.

The present note is devoted to an example in support of an idea put forward in [8]: that derivatives of $\rho(A)$ with respect to parameters can be meaningfully defined in spite of discontinuities. An ambitious project would be to have Taylor expansions on a large set $\Sigma$ of parameter values and, using a theorem of Whitney [10], to connect these expansions by a function extrapolating $\rho(A)$ smoothly outside of $\Sigma$. In a different dynamical situation, that of KAM tori, a smooth extension à la Whitney has been achieved by Chierchia and Gallavotti [3], and Pöschel [4].

In our study we have considered only a rather special set $\Sigma$ consisting of maps satisfying a Markov property. (Reference [1] should be consulted for a discussion of the poles encountered in the study of a Markovian map $f$). Note that the studies of a.c.i.m. for maps of the interval, and of SRB measures for Hénon-like maps, are typically based on perturbations of a map satisfying a Markov property (for the use of slightly more general Misiurewicz-type maps see [9], which also gives references to earlier work).

The function $\Psi(\lambda)$ that we have encountered is related to the *susceptibility* $\omega \mapsto \Psi(e^{i\omega})$ giving the response of a system to a periodic perturbation. The existence of a holomorphic extension of the susceptibility to the upper half complex plane is expected to follow from *causality* (causality says that cause preceeds effect, resulting in a *response function* $\kappa$ having support on the positive half real axis, and its Fourier transform $\hat{\kappa}$ extending holomorphically to the upper half complex plane). A discussion of nonequilibrium statistical mechanics [7] shows that the expected support and holomorphy properties hold close to equilibrium, or if uniform hyperbolicity holds. In the example discussed in this note, $\kappa$ has the right support property, but increases exponentially at infinity, and holomorphy in the upper half plane fails, corresponding the existence of a pole of $\Psi$ at $\lambda = 1/\sqrt{f'(-1)}$. This might be expressed by saying that $\rho$ is *not linearly stable.* The physically interesting situation of *large systems* (thermodynamic limit) remains quite unclear at this point.

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